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Wahlquist's metric versus an approximate solution with the same equation of state

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We compare an approximation of Wahlquist's exact solution with a stationary and axisymmetric metric for a rigidly rotating perfect fluid with the equation of state $\mu + 3p = \mu_0$, a sub-case of a global approximate metric obtained recently for some of us. This comparison allows us in particular to determine the approximate relation between the rotation velocity of the fluid and the r_0 parameter of Wahlquist's metric. Through some coordinate changes we manage to make every component of both approximate metrics equal. In this situation, the free constants of our metric take the values needed for it to be of Petrov type D. These values do not allow matching with an asymptotically flat vacuum exterior.

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1. Introduction

There are a few exact solutions of the Einstein equations describing the gravitational field inside a stationary and axisymmetric rotating perfect fluid, the basic candidates to form a stellar model in General Relativity [1, 2]. Among them, only one is known to admit a closed surface of zero pressure, the key component to build a stellar model matching the interior (source) spacetime with a suitable asymptotically flat exterior. It is the Wahlquist metric, that describes a rigidly rotating perfect fluid with a 4-velocity expressible as linear combination of the Killing vectors, possesses the mass density-pressure equation of state (EOS) $\mu + 3p = \mu_0$ and has Petrov type D [3]. Nevertheless, it has been shown in several different ways that it can not correspond to an isolated object nor be matched with an asymptotically flat exterior [3–5]. Accordingly, General Relativity still lacks of any exact solution that can describe the interior of such stellar model.

To find these global models, numerical methods and analytic approximations have therefore been the way to go. Some of us introduced a new analytic approximation scheme in [6, 7] focused in this kind of problem. It is a double approximation. The first one is post-Minkowskian with associated parameter λ, which is related with the strength of the gravitational field and the second one is a slow rotation approximation with parameter Ω, measuring the deformation of the matching surface due to the rotation of the fluid. We applied this scheme to find an approximate global solution to the gravitational field

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of a fluid with simple barotropic EOS in some cases. We found solutions for constant density in [7], for a polytropic fluid (see [8]) and more recently for a lineal equation of state $\mu + (1-n)p = \mu_0$, [9–11] (hereinafter CGMR). When $n = -2$ this last case leads to the EOS of the Wahlquist solution.

It is worth noting that despite Wahlquist's inherent interest as an exact solution, our metric is in some sense more general even fixing $n = -2$. This comes from the fact that although Wahlquist is the most general Petrov type D solution for this EOS, symmetries and motion of the fluid [12], our metric can also be of type I. More interestingly, our solution, unlike Wahlquist, can be matched to an asymptotically flat exterior and thus can describe a compact isolated object.

Another problem of Wahlquist's solution is its rotation parameter. Since Wahlquist is written in a co-rotating coordinate system, the explicit relation between the timelike Killing field and the rotation is not known. All that is known is that letting the Wahlquist's r_0 parameter go to zero in a limiting procedure (that involves a coordinate change that is singular when $r_0 \to 0$) we get the static spherically symmetric Whittaker solution [2, 3]. Despite this singular character of this limit, the relation it shows between Whittaker and Wahlquist seems to be a sound one since in the slow rotation formalism of [13], the Whittaker-like and Wahlquist-like slowly rotating metrics coincide [14, 15].

The question we try to solve in this paper is whether or not we can include an appropriate approximation of the Wahlquist solution in our family of approximate solutions. There are two ways to answer. One of them is asking our $n = -2$ solution to be of Petrov type D, since a metric with its characteristics and this Petrov type must belong to the Wahlquist family [12]. The other one is finding a coordinate change to make them to coincide.

Regarding the first one, we have already verified that our solution can take Petrov type D in [10, 11, 16]. Some of its free constants —the ones unrelated with gauge at first order— are then fixed and we found their values do not coincide with the ones they are forced to take when we matched our interior solution with an asymptotically flat exterior solution, as one expects.

The coordinate change way involves writing our solution in a co-rotating frame, and then making a series expansion of Wahlquist's solution with μ_0 as post-Minkowskian parameter —i. e. when this parameter equals zero the Wahlquist solution becomes Minkowski's metric, the same behaviour parameter λ has in our solution—. This way is more meaningful since, in spite of any result we can get from our approximate metric alone, there is always the question of whether our solution really corresponds to a parametric expansion of an exact metric. Working this way we verify explicitly this correspondence.

In Section 2 we give some notation and definitions used along the paper and we write the CGMR metric for $n = -2$ and perform the rotation to write our metric in a co-rotating coordinate system. In Section 3 we give the Wahlquist metric, write it in spheroidal-like coordinates and then write the approximate post-Minkowskian Wahlquist metric. Finally, in Section 4 we compare both solutions and determine the value of our constants and the relation between r_0 and the rotation parameters.

2. The approximate interior metric

We work within the analytical approximation scheme developed in [6, 7] and [11]. It allows to build an approximate stationary and axisymmetric solution of the Einstein's equations for a source spacetime and an asymptotically flat vacuum region around it, although in this paper we put the focus on the interior, source spacetime. This section is devoted to a brief review of its main points in the general formulation.

Let's choose t and φ to be the coordinates adapted to the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, and r and θ to the orthogonal two-dimensional surfaces [17, 18], then we can write the interior –and exterior– metric with the structure

$$\begin{aligned} \mathbf{g} = {} & \gamma_{tt}\, \boldsymbol{\omega}^t \otimes \boldsymbol{\omega}^t + \gamma_{t\varphi}(\boldsymbol{\omega}^t \otimes \boldsymbol{\omega}^\varphi + \boldsymbol{\omega}^\varphi \otimes \boldsymbol{\omega}^t) + \gamma_{\varphi\varphi}\, \boldsymbol{\omega}^\varphi \otimes \boldsymbol{\omega}^\varphi \\ & + \gamma_{rr}\, \boldsymbol{\omega}^r \otimes \boldsymbol{\omega}^r + \gamma_{r\theta}(\boldsymbol{\omega}^r \otimes \boldsymbol{\omega}^\theta + \boldsymbol{\omega}^\theta \otimes \boldsymbol{\omega}^r) + \gamma_{\theta\theta}\, \boldsymbol{\omega}^\theta \otimes \boldsymbol{\omega}^\theta \end{aligned} \tag{1}$$

in the associated cobasis $\boldsymbol{\omega}^t = dt$, $\boldsymbol{\omega}^r = dr$, $\boldsymbol{\omega}^\theta = r\,d\theta$, $\boldsymbol{\omega}^\varphi = r\sin\theta\,d\varphi$. We require these coordinates to be spherical associated to harmonic Cartesian-like ones.

The non convective 4-velocity can be written as $\boldsymbol{u} = \psi(\boldsymbol{\xi} + \omega\boldsymbol{\zeta})$. Here, ψ is a normalization factor and ω the angular velocity of the fluid. The explicit expressions for μ and p for the linear EOS with $n = -2$ are

$$p = \frac{\mu_0}{2}\left(1 - \frac{\psi_\Sigma^2}{\psi^2}\right), \quad \mu = \frac{\mu_0}{2}\left(3\frac{\psi_\Sigma^2}{\psi^2} - 1\right) \tag{2}$$

with ψ_Σ the value of ψ on the $p = 0$ surface.

For $n = -2$ the metric CGMR obtained in [11] is the following (we have made the notation change: $r_0 \to r_s$ to avoid confusion with the r_0 parameter that appears in the usual expressions of the Wahlquist metric):

$$\begin{aligned} \gamma_{rr}^{\mathrm{CGMR}} = {} & 1 + \lambda\left[m_0 - \frac{r^2}{r_s^2}\left(1 - m_2\Omega^2 P_2\right)\right] \\ & + \frac{2\lambda^2}{5}\frac{r^2}{r_s^2}\left\{m_0 - 12S - 4m_0 m_2\Omega^2 P_2 - \frac{r^2}{r_s^2}\left[\Omega^2\left(\frac{5}{3}P_2 - \frac{8}{7}\right) + \frac{1}{7}\right]\right\} \\ & + \mathcal{O}(\lambda^3,\, \Omega^4), \end{aligned} \tag{3}$$

$$\gamma_{r\theta}^{\mathrm{CGMR}} = -\lambda^2\Omega^2\frac{r^2}{r_s^2}P_2^1\left[\frac{1}{5}m_0 m_2 + \frac{1}{63}\frac{r^2}{r_s^2}(1 - 6m_2)\right] + \mathcal{O}(\lambda^3,\, \Omega^4), \tag{4}$$

$$\begin{aligned} \gamma_{\theta\theta}^{\mathrm{CGMR}} = {} & 1 + \lambda\left[m_0 - \frac{r^2}{r_s^2}\left(1 - m_2\Omega^2 P_2\right)\right] \\ & + \lambda^2\frac{r^2}{r_s^2}\left\{-\frac{1}{5}\left[18S + m_0 + 2m_0 m_2\Omega^2(2P_2 - 1)\right]\right. \\ & \qquad \left. + \frac{1}{7}\frac{r^2}{r_s^2}\left[\frac{8}{5} - \frac{\Omega^2}{3}\left(\frac{m_2}{2} - \frac{134}{15} + \left(\frac{31}{3} + \frac{23m_2}{2}\right)P_2\right)\right]\right\} \end{aligned}$$

$$+\mathcal{O}(\lambda^3,\,\Omega^4), \tag{5}$$

$$\gamma_{\varphi\varphi}^{\rm CGMR} = 1+\lambda\left[m_0-\frac{r^2}{r_s^2}\left(1-m_2\Omega^2P_2\right)\right]+\lambda^2\frac{r^2}{r_s^2}\left\{-\frac{1}{5}\left(18S+m_0+2m_0m_2\Omega^2\right)\right.$$
$$\left.+\frac{1}{7}\frac{r^2}{r_s^2}\left[\frac{8}{5}+\frac{\Omega^2}{3}\left(\frac{m_2}{2}-\frac{26}{15}+\left(\frac{1}{3}-\frac{25m_2}{2}\right)P_2\right)\right]\right\}+\mathcal{O}(\lambda^3,\,\Omega^4), \tag{6}$$

$$\gamma_{t\varphi}^{\rm CGMR} = \lambda^{3/2}\Omega\frac{r}{r_s}\left[\left(j_1-\frac{6}{5}\frac{r^2}{r_s^2}\right)P_1^1+j_3\Omega^2\frac{r^2}{r_s^2}P_3^1\right]$$
$$+\lambda^{5/2}\Omega\frac{r^3}{r_s^3}\left\{\frac{1}{5}\left[\left(j_1-12S-6m_0-m_2j_1\Omega^2\right)+\frac{6}{7}\frac{r^2}{r_s^2}\left(2+m_2\Omega^2\right)\right]P_1^1\right.$$
$$\left.+\frac{1}{3}\Omega^2\frac{r^2}{r_s^2}\left(\frac{j_3}{3}-\frac{2m_2}{5}\right)P_3^1\right\}+\mathcal{O}(\lambda^3,\,\Omega^4), \tag{7}$$

$$\gamma_{tt}^{\rm CGMR} = -1+\lambda\left[m_0-\frac{r^2}{r_s^2}\left(1-m_2\Omega^2P_2\right)\right]-\lambda^2\frac{r^4}{r_s^4}\left[\frac{1}{5}\left(1+2\Omega^2\right)-\frac{4}{7}\Omega^2(1+m_2)P_2\right]$$
$$+\mathcal{O}(\lambda^3,\,\Omega^4), \tag{8}$$

where, to simplify we have taken the gauge constants $a_0=0$, $a_2=0$, $b_2=0$ because they are not needed hereafter. The approximation parameters have the expressions

$$\Omega=\omega r_s\lambda^{-1/2},\quad \lambda=\frac{\mu_0 r_s^2}{6},\quad (8\pi G=1) \tag{9}$$

and S appears in the series expansion of Ψ_Σ, i. e., $\Psi_\Sigma=1+\lambda S$. Its worth noting Ω gives a measure of the deformation of the source due to its rotation. This solution is apparently less interesting than the Wahlquist exact solution for the same kind of source because it is an approximation. Nevertheless, it is more general in a sense because it is a Petrov type I solution unless

$$m_2=\frac{6}{5}+\mathcal{O}(\lambda,\,\Omega^2),\quad j_3=\frac{36}{175}+\mathcal{O}(\lambda,\,\Omega^2), \tag{10}$$

in which case it becomes a Petrov type D solution [11]. Moreover, in our solution the parameter ω has the clear meaning of angular velocity of the fluid,

$$\omega=\frac{u^\varphi}{u^t} \tag{11}$$

and its vanishing leads to a static solution (i. e. $\gamma_{t\varphi}=0$) while there is no equivalent parameter in Wahlquist. Another interesting property of our solution is that, in general, it can be matched to an exterior asymptotically flat solution which is not the case for Wahlquist metric.

If we want to compare this approximate solution with Wahlquist's metric we have to start finding their expressions in the same coordinates. The first problem is that Wahlquist metric is written in a co–rotating coordinate system and CGMR is not, so first we must

choose between the two kinds of coordinates. Changing CGMR to a co–rotating system is straightforward doing

$$\varphi \to \varphi + \frac{\lambda^{1/2}\Omega}{r_s}t, \quad t \to t \tag{12}$$

and then in the co–rotating system the metric components are:

$$\begin{aligned}
\gamma_{tt}^{\rm CGMR} &= -1 + \lambda \left\{ m_0 + \frac{r^2}{r_s^2}\left[-1 + \frac{1}{3}\Omega^2\left(2 + (3m_2 - 2)P_2\right)\right]\right\} \\
&+ \lambda^2 \frac{r^2}{r_s^2}\left\{\frac{2}{3}\Omega^2(2j_1 - m_0)(P_2 - 1) - \frac{1}{5}\frac{r^2}{r_s^2}\left[1 - \frac{2}{3}\Omega^2\left(4 + \frac{1}{7}(30m_2 - 19)P_2\right)\right]\right\} \\
&+ \mathcal{O}(\lambda^3, \Omega^4),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\gamma_{t\varphi}^{\rm CGMR} &= -\Omega\lambda^{1/2}\frac{r}{r_s}\left\{P_1^1 + \lambda\left[\left(m_0 - j_1 + \frac{1}{5}\frac{r^2}{r_s^2}\left(1 - \Omega^2 m_2\right)\right)P_1^1\right.\right. \\
&\left. - \Omega^2\frac{r^2}{r_s^2}\left(j_3 - \frac{m_2}{5}\right)P_3^1\right] + \lambda^2\frac{r^2}{r_s^2}\left[m_0 - \frac{j_1}{5} - \frac{6}{5}S - \frac{4}{35}\frac{r^2}{r_s^2}\right. \\
&+ \Omega^2\left(\left(\frac{m_2}{5}(j_1 - 2m_0) - \frac{1}{35}\frac{r^2}{r_s^2}(m_2 + 3)\right)P_1^1\right. \\
&\left.\left.\left. - \frac{r^2}{r_s^2}\left(\frac{j_3}{9} - \frac{1}{315} - \frac{m_2}{70}\right)P_3^1\right)\right]\right\} + \mathcal{O}(\lambda^3, \Omega^4)
\end{aligned} \tag{14}$$

and the other components remain unchanged. Let us remark that the $\gamma_{t\varphi}$ component is now of order $\lambda^{1/2}$ instead of the order $\lambda^{3/2}$ it was in the original coordinates (see [11] for some comments).

3. The Wahlquist metric

The next steps in the comparison are, using the singularity free Wahlquist metric, first expand it in the appropriate approximation parameters and then make coordinate changes to reduce it to a particular case of CGMR.

We start writing the 4-velocity of the fluid as

$$\vec{u} = f^{-1/2}\,\partial_t \qquad (g_{\alpha\beta}u^\alpha u^\beta = -1)\,, \tag{15}$$

the energy density and pressure are given by

$$\left\{\begin{aligned} \mu &= \frac{1}{2}\mu_0(3b^2 f - 1) \\ p &= \frac{1}{2}\mu_0(1 - b^2 f) \end{aligned}\right. , \tag{16}$$

and the singularity free Wahlquist metric reads [2]

$$ds^2 = -f(dt + A\,d\varphi)^2 +$$

$$r_0^2(\xi^2+\eta^2)\left[\frac{c^2h_1h_2}{h_1-h_2}\,d\varphi^2+\frac{d\xi^2}{(1-k^2\xi^2)h_1}+\frac{d\eta^2}{(1+k^2\eta^2)h_2}\right] \tag{17}$$

where

$$f(\xi,\eta)=\frac{h_1-h_2}{\xi^2+\eta^2},\qquad A=c\,r_0\left(\frac{\xi^2h_2+\eta^2h_1}{h_1-h_2}-\eta_0^2\right) \tag{18}$$

$$h_1(\xi)=1+\xi^2+\frac{\xi}{b^2}\left[\xi-\frac{1}{k}(1-k^2\xi^2)^{1/2}\arcsin(k\,\xi)\right] \tag{19}$$

$$h_2(\eta)=1-\eta^2-\frac{\eta}{b^2}\left[\eta-\frac{1}{k}(1+k^2\eta^2)^{1/2}\mathrm{arcsinh}(k\,\eta)\right] \tag{20}$$

and

$$k^2\equiv\frac{1}{2}\,\mu_0\,r_0^2b^2\,. \tag{21}$$

Here μ_0, b, r_0 are arbitrary constants and η_0 and c are adjusted so that the solution behaves properly on the axis. The symmetry axis is located at $\eta=\eta_0$ where

$$h_2(\eta_0)=0\,, \tag{22}$$

and to satisfy the regularity condition c must be

$$\frac{1}{c}=\frac{1}{2}(1+k^2\eta_0^2)^{1/2}\left.\frac{dh_2}{d\eta}\right|_{\eta=\eta_0}. \tag{23}$$

Therefore η_0 and c become functions of the constants μ_0 and b.

The constants b and μ_0 are the values of the normalization factor $f^{-1/2}$ and the energy density on the matching surface of zero pressure. The r_0 parameter allows to get Wahlquist's static limit —Whittaker's metric [19]— making the change

$$\{\xi,\,\eta\}\to\{R,\chi\}:\ \{\xi=\frac{R}{r_0},\eta=\cos\chi\} \tag{24}$$

and letting r_0 go to zero [3]. This suggests a connection between r_0 and Wahlquist's fluid rotation speed, although it is by no means necessarily true. Actually, to the best of our knowledge, there is no proved explicit expression for such rotation speed. The parameters $\{r_0,\,\mu_0\}$ will be the natural choice for us to expand Wahlquist metric, but first we must find the change to spherical-like coordinates.

3.1. Wahlquist written in spherical–like coordinates

Our approximate metric (3) to (6), (13) and (14) is written in "standard" spherical coordinates —in the sense that in when $\lambda=0$ they are the usual ones— so we need to

find a consistent way to write the Wahlquist metric in a set of coordinates as close to ours as possible to begin with.

In this regard we note first that if we put $\mu_0 = 0$ in the Wahlquist metric (17) we obtain Minkowski in oblate spheroidal coordinates $\{\xi, \eta\}$, whose coordinate lines are oblate confocal ellipses and confocal orthogonal hyperbolas. From these coordinates is easy to go to Kepler coordinates $\{R, \chi\}$ changing $\xi = R/r_0, \eta = \cos\chi$, where R represents the semi-minor axis of the ellipses, χ the Kepler eccentric polar angle and r_0 the focal length. Finally we get standard spherical coordinates $\{r, \theta\}$ by changing

$$\sqrt{R^2 + r_0^2}\sin\chi = r\sin\theta, \quad R\cos\chi = r\cos\theta. \tag{25}$$

Moreover, the limiting procedure (24) from Wahlquist to its static limit (Whittaker) has a similar form, in this case leading to Kepler-like coordinates.

These considerations suggest to look for a change of coordinates in the Wahlquist metric (prior to any limit) so that the new coordinates "directly represent" spheroidal–like coordinates. Fortunately this approach arises when we plot the graphics of $h_1(\xi)$ and $h_2(\eta)$ (Fig. 1.) We can see that these curves describe locally a hyperbolic cosine and a squared sine, respectively. Taking all this into account we write

$$\{\xi, \eta\} \to \{R,\, \chi\} : \quad \begin{cases} h_1(\xi) = 1 + \dfrac{R^2}{r_0^2} = 1 + \dfrac{R_1^2}{r_0^2} \\ 1 - h_2(\eta) = \cos^2\chi = \dfrac{R_2^2}{r_0^2}\,. \end{cases} \tag{26}$$

where we introduce R_1, R_2 just to simplify calculations later. Let us write now the two dimensional metric spanned by $\{\xi, \eta\}$

$$d\Sigma^2 = Ad\xi^2 + Bd\eta^2 \tag{27}$$

in terms of $\{R_1,\, R_2\}$. Since

$$d\xi = \frac{dh_1}{dh_1/d\xi} \quad \text{and} \quad d\eta = \frac{dh_2}{dh_2/d\eta}\,, \tag{28}$$

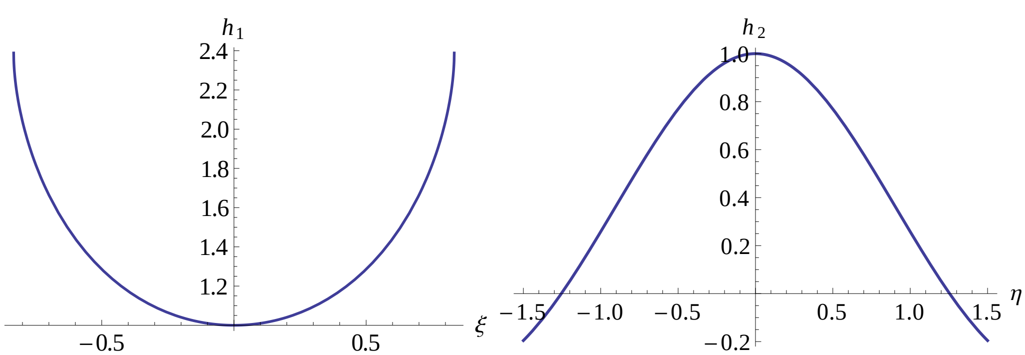

FIG. 1. Behaviour of the $h_1(\xi)$ and $h_2(\eta)$ functions for certain values of k and b.

taking h_1, h_2 as functions of R_1 and R_2

$$dh_1 = \frac{2R_1}{r_0^2} dR_1 \,, \quad -dh_2 = \frac{2R_2}{r_0^2} dR_2 \,, \tag{29}$$

we get

$$d\Sigma^2 = m_{11} \left(\frac{2R_1}{r_0^2} dR_1 \right)^2 + m_{22} \left(\frac{2R_2}{r_0^2} dR_2 \right)^2 , \tag{30}$$

where

$$m_{11} = \frac{A}{(dh_1/d\xi)^2}, \quad m_{22} = \frac{B}{(dh_2/d\eta)^2}. \tag{31}$$

Now let us do another coordinate change to a kind of spherical coordinates $\{r, \theta\}$ using the previous relations Eqs. (25) and (26). Then, R_1 and R_2 are the following functions of r and θ

$$\{R_1, R_2\} \to \{r, \theta\} : \quad \begin{cases} R_1 = \sqrt{\dfrac{1}{2} \left(r^2 - r_0^2 + \sqrt{(r^2 - r_0^2)^2 + 4r^2 r_0^2 \cos^2\theta} \right)} \,, \\ R_2 = \sqrt{\dfrac{1}{2} \left(r_0^2 - r^2 + \sqrt{(r^2 - r_0^2)^2 + 4r^2 r_0^2 \cos^2\theta} \right)} \,, \end{cases} \tag{32}$$

and if we define the function

$$F = (r^2 - r_0^2)^2 + 4r^2 r_0^2 \cos^2\theta \,,$$

we have that the metric (27) in $\{r, \theta\}$ coordinates

$$d\Sigma^2 = g_{rr} dr^2 + 2g_{r\theta} dr d\theta + g_{\theta\theta} d\theta^2$$

has the coefficients

$$\begin{aligned} g_{rr} = \frac{2r^2}{r_0^4 F} \Big[& \sqrt{F} (r^2 - r_0^2 + 2r_0^2 \cos^2\theta)(m_{11} - m_{22}) \\ & + (F - 2r_0^4 \sin^2\theta \cos^2\theta)(m_{11} + m_{22}) \Big] \,, \end{aligned} \tag{33}$$

$$\begin{aligned} g_{r\theta} = -\sin\theta\cos\theta \frac{2r^3}{r_0^2 F} \Big[& \sqrt{F}\, (m_{11} - m_{22}) \\ & + \left(r^2 - r_0^2 + 2r_0^2 \cos^2\theta \right) (m_{11} + m_{22}) \Big] \,, \end{aligned} \tag{34}$$

$$g_{\theta\theta} = \sin^2\theta \cos^2\theta \frac{4r^4}{F} (m_{11} + m_{22}). \tag{35}$$

Only the terms $m_{11} + m_{22}$ and $m_{11} - m_{22}$ depend on both $\{\mu_0, r_0\}$; the remaining terms depend on r_0 alone.

Now we are going to write the full metric in terms of $\{r, \theta\}$; notice that the inversion can only be approximately done (in a series of μ_0). First, we determine η_0 up to order μ_0^3, and then c to the same order. This last series depend on b^2 and we want to compare with our approximation, where the pressure is of a higher order in the post-Minkowskian parameter (in this case in μ_0) than the density. To determine how b depends on μ_0 we recall that for $\mu_0 = 0$ the Wahlquist metric becomes Minkowski's metric written in oblate spheroidal coordinates. From Eq. (16) this means that

$$f(\mu_0 = 0) = 1 \quad \text{and} \quad p = \frac{1}{2}\mu_0 \left\{1 - b^2[1 + \mathcal{O}(\mu_0)]\right\} , \tag{36}$$

so if we want to have $p = p_1\mu_0^2 + \mathcal{O}[\mu_0^3]$, the constant must be $b^2 = 1 + \mathcal{O}[\mu_0]$. Accordingly, we are going to assume

$$b^2 = 1 + \frac{1}{3}\mu_0\sigma_1 + \mu_0^2\sigma_2 + \mathcal{O}(\mu_0^3) \tag{37}$$

where σ_1 and σ_2 are two new constants. Using this hypothesis on b^2 we obtain for the constants η_0 and c up to $\mathcal{O}(\mu_0^3)$

$$\eta_0 = 1 + \frac{1}{12}\mu_0 r_0^2 \left\{1 + \frac{1}{120}\mu_0 r_0^2 \left[11 + \mu_0 \left(\frac{73}{28}r_0^2 - 8\sigma_1\right)\right]\right\} + \mathcal{O}(\mu_0^3), \tag{38}$$

$$c = -1 + \frac{1}{12}r_0^2\mu_0^2 \left[\sigma_1 - \frac{r_0^2}{3} + \mu_0 \left(3\sigma_2 - \frac{r_0^4}{30}\right)\right] + \mathcal{O}(\mu_0^3). \tag{39}$$

Next, we invert the change of coordinates Eq. (26), what gives

$$\xi^2 = \frac{R_1^2}{r_0^2}\left(1 - \frac{1}{6}\mu_0 R_1^2 \left\{1 - \frac{1}{15}\mu_0 R_1^2 \left[2 - \mu_0 \left(\sigma_1 + \frac{37}{84}R_1^2\right)\right]\right\}\right) + \mathcal{O}(\mu_0^4) \tag{40}$$

$$\eta^2 = \frac{R_2^2}{r_0^2}\left(1 + \frac{1}{6}\mu_0 R_2^2 \left\{1 + \frac{1}{15}\mu_0 R_2^2 \left[2 - \mu_0 \left(\sigma_1 - \frac{37}{84}R_2^2\right)\right]\right\}\right) + \mathcal{O}(\mu_0^4) \tag{41}$$

And finally, by doing the coordinate change $\{R_1, R_2\} \to \{r, \theta\}$ we obtain the metric coefficients up to $\mathcal{O}(\mu_0^3)$ in the spherical–like coordinates desired

$$\begin{aligned}\gamma_{rr}^{\mathrm{W}} = 1 &+ \frac{\mu_0}{6}(r_0^2 - r^2) + \frac{\mu_0^2}{6}\left[\sigma_1(r^2 - r_0^2\sin^2\theta) + r_0^2\left(\frac{4r^2}{5} - \frac{r_0^2}{3}\right)\cos^2\theta\right.\\ &\left. + \frac{7}{15}(r_0^2 - r^2)^2\right] + \frac{\mu_0^3}{90}\left\{\frac{\sigma_1}{2}\left[r_0^2\left(5r_0^2 - 7r^2\right)\cos^2\theta - 7(r_0^2 - r^2)^2\right]\right.\\ &+ 45\sigma_2(r^2 - r_0^2\sin^2\theta) + \frac{r_0^2}{21}(r_0^2 - r^2)(85r^2 - 28r_0^2)\cos^2\theta\\ &\left. + \frac{149}{84}(r_0^2 - r^2)^3 - \frac{r_0^4}{2}r^2\cos^4\theta\right\}\end{aligned} \tag{42}$$

$$\begin{aligned}\gamma^{\mathrm{W}}_{\theta\theta} &= 1+\frac{\mu_0}{6}(r_0^2-r^2)+\frac{\mu_0^2}{9}\left[r_0^2\left(\frac{r^2}{5}+\frac{r_0^2}{2}-\frac{3}{2}\sigma_1\right)\cos^2\theta+\frac{1}{5}(r^2-r_0^2)^2\right]\\
&\quad+\frac{\mu_0^3}{180}\left\{\sigma_1\left[r_0^2(3r^2-5r_0^2)\cos^2\theta-2(r_0^2-r^2)^2\right]-90\sigma_2 r_0^2\cos^2\theta\right.\\
&\quad\left.+\frac{r_0^2}{21}(r_0^2-r^2)(37r^2+56r_0^2)\cos^2\theta+r_0^4r^2\cos^4\theta+\frac{37}{42}(r_0^2-r^2)^3\right\}\end{aligned}\tag{43}$$

$$\begin{aligned}\gamma^{\mathrm{W}}_{r\theta} &= \frac{\mu_0^2r_0^2}{18}\sin\theta\cos\theta\left\{r_0^2-r^2-3\sigma_1\right.\\
&\quad\left.-\frac{\mu_0}{10}\left[5\sigma_1(r_0^2-r^2)+90\sigma_2-r_0^2r^2\cos\theta^2-\frac{8}{3}(r_0^2-r^2)^2\right]\right\}\end{aligned}\tag{44}$$

$$\begin{aligned}\gamma^{\mathrm{W}}_{\varphi\varphi} &= 1+\frac{\mu_0}{6}(r_0^2-r^2)-\frac{\mu_0^2}{6}\left\{r_0^2\left[\sigma_1-\frac{1}{15}\left(7r_0^2-\frac{13}{2}r^2\right)\right]-\frac{3}{10}r_0^2r^2\cos^2\theta\right.\\
&\quad\left.-\frac{2}{15}r^4\right\}+\frac{\mu_0^3}{90}\left\{\sigma_1\left[\frac{r_0^2}{2}(9r^2-7r_0^2)-r_0^2r^2\cos^2\theta-r^4\right]-45r_0^2\sigma_2\right.\\
&\quad\left.+\frac{149}{84}r_0^6-\frac{43}{14}r_0^4r^2+\frac{11}{7}r_0^2r^4-\frac{37}{84}r^6-\frac{1}{84}r_0^2r^2(95r^2-151r_0^2)\cos^2\theta\right\}\end{aligned}\tag{45}$$

$$\begin{aligned}\gamma^{\mathrm{W}}_{t\varphi} &= -\frac{\mu_0r_0}{6}r\sin\theta\left\{1+\frac{\mu_0}{30}(r^2+3r_0^2)+\frac{\mu_0^2}{15}\left[\sigma_1\left(r^2-\frac{13}{4}r_0^2\right)\right.\right.\\
&\quad\left.\left.+\frac{r_0^2}{84}(97r_0^2-41r^2)+\frac{4}{21}r^4+\frac{3}{28}r_0^2r^2\cos^2\theta\right]\right\}\end{aligned}\tag{46}$$

$$\begin{aligned}\gamma^{\mathrm{W}}_{tt} &= -1+\frac{\mu_0}{6}(r_0^2-r^2)-\frac{\mu_0^2}{180}\left\{(r_0^2-r^2)^2-4r^2r_0^2\cos^2\theta\right\}+\frac{\mu_0^3}{90}\left\{(r_0^2-r^2)\times\right.\\
&\quad\left.\times\left[\frac{4}{21}(r_0^2-r^2)^2+\frac{3}{14}r^2r_0^2\cos^2\theta\right]-\sigma_1\left[(r^2-r_0^2)^2+r^2r_0^2\cos^2\theta\right]\right\}\end{aligned}\tag{47}$$

4. Comparing the approximate Wahlquist solution with the CGMR solution in co–rotating coordinates

Now we face the problem of identification of the parameters and to perform the final adjustments of coordinates needed to make every term in Wahlquist and CGMR equal.

To get an idea of the problems arising, we analyze first the static limit. By using the relation between mass parameters $\lambda=\frac{1}{6}\mu_0r_s^2$ and making $r_0=0$ in Eq. (42) we obtain the expression for the γ_{rr} coefficient of the static metric

$$\gamma^{\mathrm{W}}_{rr}(r_0=0)=1-\frac{r^2\lambda}{{r_s}^2}+\frac{2r^2\lambda^2\left(7r^2+15\sigma_1\right)}{5{r_s}^4}+\mathcal{O}(\lambda^3)\tag{48}$$

and upon comparison with the corresponding static limit of CGMR

$$\gamma_{rr}^{\mathrm{CGMR}}(\Omega=0) = 1 + m_0\lambda - \frac{r^2\lambda}{{r_s}^2} - \frac{2r^4\lambda^2}{35{r_s}^4} + \frac{2m_0 r^2\lambda^2}{5{r_s}^2} - \frac{24r^2S\lambda^2}{5{r_s}^2} + \mathcal{O}(\lambda^3) \tag{49}$$

we can see that there are discrepancies among r^4 terms. To some extent this was to be expected since CGMR was written in coordinates associated to harmonic ones and no such a condition has been imposed on Wahlquist. In this particular case, the two metrics can be rendered exactly equal with a change of the form $r \to r\big[1+\lambda^2 f(r)\big]$.

4.1. Adjusting parameters

We go back now to the non–static case. If we compare the lowest order term in $g_{t\varphi}$ and g_{tt} of both solutions we can see that the constants λ and Ω of the CGMR solution must be related with the μ_0 and r_0 constants of Wahlquist as follow

$$\mu_0 = \frac{6\lambda}{r_s^2}, \quad r_0 = -\frac{\kappa\, r_s\Omega}{\lambda^{1/2}} \tag{50}$$

where κ gives the proportionality between the rotation parameters of both solutions, r_0 and Ω, and will be determined later on. If we perform this identification we get to a new difficulty because the Wahlquist solution has λ-free terms with Ω dependence. These terms appear associated with powers of $\mu_0 r_0^2$ $\big[$or $\kappa^2\Omega^2$ using Eq. (50)$\big]$. This is not possible in our self–gravitating solution building scheme. This issue can be solved using the remaining freedom in time scale and $\{r,\theta\}$ coordinates. The changes we can do are

$$t = T\left(1+\mu_0 F_1 + \mu_0^2 F_2 + \cdots\right) \tag{51}$$

$$\begin{aligned} r &= R\big[1+\mu_0 G_1(R,\Theta) + \mu_0^2 G_2(R,\Theta) + \cdots\big] \\ \theta &= \Theta + \mu_0 \sin\Theta\big[H_1(R,\Theta) + \mu_0 H_2(R,\Theta) + \cdots\big]\,, \end{aligned} \tag{52}$$

with F_i constants depending on the parameters and G_i, H_i undetermined functions. Imposing vanishing of these unwanted terms, we get the time scale change

$$t = T\left\{1+\frac{\mu_0 r_0^2}{12}\left[1+\frac{11\mu_0 r_0^2}{120}\left(1+\frac{73\mu_0 r_0^2}{308}\right)\right]\right\} + \mathcal{O}(\mu_0^4) \tag{53}$$

and the $\{r,\,\theta\}$ changes

$$r = R\left\{1-\frac{\mu_0 r_0^2}{12}\left(1+\frac{\mu_0 r_0^2}{3}\left[\frac{41}{40}-\cos^2\Theta+\frac{\mu_0 r_0^2}{60}\left(\frac{191}{56}-\cos^2\Theta\right)\right]\right)\right\} + \mathcal{O}(\mu_0^4), \tag{54}$$

$$\theta = \Theta - \frac{\mu_0^2 r_0^4}{36}\sin\Theta\cos\Theta\left(1+\frac{\mu_0 r_0^2}{10}\right) + \mathcal{O}(\mu_0^4). \tag{55}$$

Note that the symmetry axis for the old coordinates is located at $\theta = 0$, π and due to the presence of the $\sin\Theta$ it remains at $\Theta = 0, \pi$. We will maintain this condition for all the coordinate changes of the θ coordinate.

Now, we introduce these changes in our last expression of Wahlquist metric obtaining up to $\mathcal{O}(\mu_0^3)$

$$\begin{aligned}\gamma^{\rm W}_{RR} &= 1 - \frac{\mu_0}{6}R^2 + \mu_0^2\left\{\frac{7}{90}R^4 + \frac{\sigma_1}{6}R^2 + r_0^2\left[\frac{R^2}{10}\left(\frac{4}{3}\cos^2\Theta - 1\right) - \frac{\sigma_1}{6}\sin^2\Theta\right]\right\}\\ &+ \mu_0^3 r_0^2\left[\frac{17}{42}R^4\left(\frac{1}{20} - \frac{1}{9}\cos^2\Theta\right) - \frac{\sigma_2}{2}\sin^2\Theta + \frac{\sigma_1}{45}R^2\left(1 - \frac{7}{4}\cos^2\Theta\right)\right], \end{aligned} \tag{56}$$

$$\gamma^{\rm W}_{R\Theta} = -\sin\Theta\cos\Theta\frac{\mu_0^2 r_0^2}{6}\left[\sigma_1 + \frac{1}{3}R^2 + \mu_0\left(3\sigma_2 - \frac{\sigma_1}{6}R^2 - \frac{4}{45}R^4\right)\right], \tag{57}$$

$$\begin{aligned}\gamma^{\rm W}_{\Theta\Theta} &= 1 - \frac{\mu_0}{6}R^2 + \frac{1}{45}\mu_0^2\left\{R^4 + r_0^2\left[R^2\left(\cos\Theta^2 + \frac{1}{2}\right) - \frac{15}{2}\sigma_1\cos^2\Theta\right]\right\}\\ &+ \frac{\mu_0^3}{2}r_0^2\left[\frac{\sigma_1}{90}R^2\left(4 + 3\cos\Theta^2\right) + \frac{1}{140}R^4\left(1 - \frac{74}{27}\cos^2\Theta\right) - \sigma_2\cos^2\Theta\right], \end{aligned} \tag{58}$$

$$\begin{aligned}\gamma^{\rm W}_{\varphi\varphi} &= 1 - \frac{\mu_0}{6}R^2 + \frac{\mu_0^2}{3}\left\{\frac{1}{15}R^4 + \frac{1}{2}r_0^2\left[\frac{R^2}{10}(3\cos\Theta^2 - 1) - \sigma_1\right]\right\}\\ &+ \mu_0^3 r_0^2\left[\frac{\sigma_1}{90}R^2\left(\frac{9}{2} - \cos^2\Theta\right) + \frac{R^4}{63}\left(\frac{2}{5} - \frac{19}{24}\cos^2\Theta\right) - \frac{\sigma_2}{2}\right], \end{aligned} \tag{59}$$

$$\begin{aligned}\gamma^{\rm W}_{t\varphi} &= -\frac{\mu_0}{6}r_0 R\sin\Theta\left\{1 + \frac{\mu_0}{30}\left(R^2 + \frac{r_0^2}{2}\right)\right.\\ &\left. + \frac{\mu_0^2}{20}r_0^2\left[\frac{R^2}{7}\left(\cos^2\Theta - \frac{103}{18}\right) - \frac{13}{3}\sigma_1\right]\right\}, \end{aligned} \tag{60}$$

$$\begin{aligned}\gamma^{\rm W}_{tt} &= -1 - \frac{\mu_0}{6}R^2 - \frac{\mu_0^2}{180}\left[R^4 - 2r_0^2 R^2\left(1 + 2\cos^2\Theta\right)\right]\\ &+ \mu_0^3\frac{r_0^2}{90}\left[\sigma_1 R^2(2 - \cos^2\Theta) + \frac{R^4}{14}\left(\frac{55}{6} - 3\cos^2\Theta\right)\right]. \end{aligned} \tag{61}$$

After dealing with μ_0 and r_0, we have to find expressions for b and κ. Recalling Eq. (37), we wrote b^2 as a series in μ_0 and σ_1, σ_2. However, according to the approximation, b^2 should also contain terms in $\mu_0 r_0^2$. We get this behaviour if we write

$$\sigma_1 \to (\sigma_1 r_s^2 + r_0^2\nu_1) \tag{62}$$

$$\sigma_2 \to (\sigma_2 r_s^2 + r_0^2\nu_2)r_s^2 \tag{63}$$

with the distance r_s added for dimensional reasons (also appears in the change of constants (50) and in the CGMR solution) which represents the radius of the non–rotating matching surface. In terms of λ and Ω this leads to

$$b^2 \approx 1 + \alpha_0\Omega^2 + \beta_1\lambda + \alpha_1\Omega^2\lambda + \beta_2\lambda^2 + \cdots \tag{64}$$

with α_i and β_i constants to be adjusted.

Now we can write the approximate Wahlquist metric in terms of our parameters λ and Ω using (50). Comparing the lower terms in λ for $g_{t\varphi}$ of the CGMR co–rotating solution and the approximate Wahlquist solution just built, we can determine the proportionality constant κ to be a series in our rotation parameter Ω

$$\kappa = 1 - \frac{\Omega^2}{10} + \mathcal{O}(\Omega^3) \tag{65}$$

4.2. Adjusting terms

Once the relations between the approximation parameters of both metrics are determined we can obtain the expression of the approximate Wahlquist metric written in the same parameters we have used for the CGMR co–rotating solution. To make both solutions coincide the remaining freedom is a change in the $\{r, \theta\}$ coordinates of the type displayed in Eq. (52). If we make this change in the Wahlquist metric

$$\begin{aligned} r \to r \Bigg\{ & 1 + \lambda\Omega^2 \left(3\sigma_1 \sin^2\theta - \frac{1}{2}\frac{r^2}{r_s^2}\cos^2\theta \right) \\ & - \lambda^2 \left[\frac{9}{5}\frac{r^2}{r_s^2}\sigma_1 + \frac{2}{7}\frac{r^4}{r_s^4} - \frac{1}{70}\Omega^2 \frac{r^4}{r_s^4} \left(\frac{13}{3} + 33\cos^2\theta \right) \right] \Bigg\} + \mathcal{O}(\lambda^3, \Omega^4), \end{aligned} \tag{66}$$

$$\theta \to \theta + \lambda\Omega^2 \sin\theta\cos\theta \left(\frac{1}{2}\frac{r^2}{r_s^2} + 3\sigma_1 - \frac{29}{210}\lambda\frac{r^4}{r_s^4} \right) + \mathcal{O}(\lambda^3, \Omega^4), \tag{67}$$

we get that, for the two metrics to be exactly equal up to $\mathcal{O}(\lambda^2, \Omega^3)$ the free parameters of the CGMR co–rotating solution must be

$$m_0 = \mathcal{O}(\lambda^2, \Omega^4), \qquad m_2 = \frac{6}{5}(1 + 2\lambda S) + \mathcal{O}(\lambda^2, \Omega^2), \tag{68}$$

$$j_1 = \frac{9}{5}\Omega^2 S + \mathcal{O}(\lambda, \Omega^4), \qquad j_3 = \frac{36}{175} + \mathcal{O}(\lambda, \Omega^2) \tag{69}$$

and the free parameters of the approximate Wahlquist solution must be

$$\sigma_1 = S, \quad \sigma_2 = 0, \quad \nu_1 = \frac{1}{2}, \quad \nu_2 = \frac{1}{18}. \tag{70}$$

This gives b^2 as

$$b^2 = (1 + \Omega^2)(1 + 2\lambda S) + \mathcal{O}(\lambda^2, \Omega^4). \tag{71}$$

This result agrees with the meaning of S and b. The term $(1+\Omega^2)$ comes from the change of the normalization factor over the transformation of the temporal coordinate Eq. (53) we have done .

The final expressions for the metric components in the orthonormal basis are, $\mathcal{O}(\lambda^3,\, \Omega^4)$

$$\gamma_{rr} = 1 - \lambda\frac{r^2}{r_s^2}\left\{1-\frac{6}{5}\Omega^2 P_2\right\} + \frac{2}{5}\lambda^2\left\{-12S - \frac{1}{7}\frac{r^2}{r_s^2} + \Omega^2\left[\frac{8}{7}\frac{r^2}{r_s^2} + \left(6S - \frac{5}{3}\frac{r^2}{r_s^2}\right)P_2\right]\right\}\frac{r^2}{r_s^2} \tag{72}$$

$$\gamma_{r\theta} = \frac{31}{315}\lambda^2\Omega^2\frac{r^4}{r_s^4}P_2^1 \tag{73}$$

$$\gamma_{\theta\theta} = 1 - \lambda\frac{r^2}{r_s^2}\left(1-\frac{6}{5}\Omega^2 P_2\right) - \frac{2}{21}\lambda^2\frac{r^2}{r_s^2}\left\{\frac{189}{5}S - \frac{12}{5}\frac{r^2}{r_s^2} + \Omega^2\left[-\frac{25}{6}\frac{r^2}{r_s^2} + \frac{1}{5}\left(\frac{181}{3}\frac{r^2}{r_s^2} - 126S\right)P_2\right]\right\} \tag{74}$$

$$\gamma_{\varphi\varphi} = 1 - \lambda\frac{r^2}{r_s^2}\left(1-\frac{6}{5}\Omega^2 P_2\right) - \frac{2}{105}\lambda^2\frac{r^2}{r_s^2}\left\{189S - 12\frac{r^2}{r_s^2} + \Omega^2\left[\frac{17}{6}\frac{r^2}{r_s^2} + \left(\frac{110}{3}\frac{r^2}{r_s^2} - 126S\right)P_2\right]\right\}, \tag{75}$$

$$\gamma_{t\varphi} = -\lambda^{1/2}\Omega\frac{r}{r_s}\left\{P_1^1 + \frac{\lambda}{5}\left[\frac{r^2}{r_s^2}P_1^1 - 3\Omega^2\left(\left(3S + \frac{2}{5}\frac{r^2}{r_s^2}\right)P_1^1 - \frac{2}{35}P_3^1\right)\right]\right\} \tag{76}$$

$$\gamma_{tt} = -1 - \lambda\frac{r^2}{r_s^2}\left[1 - \frac{2}{3}\Omega^2\left(1+\frac{4}{5}P_2\right)\right] + \frac{\lambda^2}{5}\frac{r^2}{r_s^2}\left\{-\frac{r^2}{r_s^2} + \Omega^2\left[\frac{8}{3}\frac{r^2}{r_s^2} + \left(12S + \frac{34}{21}\frac{r^2}{r_s^2}\right)P_2\right]\right\} \tag{77}$$

To verify the whole procedure we can compare now with the conditions necessary for our $n=-2$ approximate metric to be of type Petrov D [11, 16, 20], i.e., Eq. (10). They are compatible with the values of the constants m_2 and j_3 we have just found in (68) and (69), as wished. Also, when matched with an asymptotically flat exterior, m_2, j_3 and the rest of the metric free constants can only have the expressions we found in [11]. Since the $n=-2$ fluid for a type D interior does not satisfy the matched expressions, we concluded then that it can not be the source of such exterior. Nevertheless, it is worth noting here that CGMR contains a $n=-2$ sub-case that lacks this problem and can indeed be matched that way. It has then all the characteristics of Wahlquist's fluid but Petrov type I instead of D.

Note, finally, that the Cartesian coordinates associated to the spherical-like coordinates used above are not harmonic. Nevertheless, since Eqs. (72) to (77) correspond as well to the stopped $n=-2$ CGMR with particular values of the free constants, undoing the change (12) they become harmonic again.

5. Remarks

In this work we have taken the singularity free Wahlquist metric and managed to transform it into the form the CGMR metric takes when written in a co–rotating coordinate system.

We have identified Wahlquist's parameters corresponding to λ and Ω of [6]. Doing this, we have found an expansion of the parameter r_0 of Wahlquist's metric in terms of our Ω. Accordingly, now we have an approximate expression of r_0 in terms of physical quantities

$$r_0 = -\frac{r_s}{\sqrt{\lambda}}\Omega\left(1-\frac{\Omega^2}{10}\right)+\mathcal{O}(\Omega^4) = -\frac{6}{\mu_0}\omega\left(1-\frac{3\omega^2}{5\mu_0}\right)+\mathcal{O}\left(\frac{\omega^4}{\mu_0^2}\right) \qquad (8\pi G=1)\,. \quad (78)$$

To the best of our knowledge its qualitative relation with the rotation was only guessed through the singular limiting procedure that takes Wahlquist and leads to Whittaker's metric but no parametrization of it in terms of well defined quantities had been given.

Last, notice that the usual interpretation of $\omega = \frac{u_\varphi}{u_t}$ as rotation speed of the fluid as seen from the infinite lacks sense if we deal with a metric that is not matched with an asymptotically flat exterior. Besides, the definition of stationarity and axisymmetry allows a change of coordinates $\{t=t',\, \varphi=\varphi'+at'\}$ that can modify the value of ω to $\omega'=\omega-a$ or make it zero (the case of co-rotating frames). Nevertheless, when dealing with a family of metrics explicitly dependent on ω, its value can be important. In the case of, e. g., CGMR, we see that written in co-rotating coordinates $u_{t'}/u_{\varphi'}=0$ but ω is part of the metric functions and actually, $\omega\to 0$ still leads to a static metric. In this sense, the characterization of r_0 (78) is meaningful.

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